Waves in strata

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The waves of practical interest in reflection seismology are usually complicated because the propagation velocities are generally complex. In this book, we have chosen to build up the complexity of the waves we consider, chapter by chapter. The simplest waves to understand are simple plane waves and spherical waves propagating through a constant-velocity medium. In seismology however, the earth's velocity is almost never well approximated by a constant. A good first approximation is to assume that the earth's velocity increases with depth. In this situation, the simple planar and circular wavefronts are modified by the effects of $v(z)$. In this chapter we study the basic equations describing plane-like and spherical-like waves propagating in media where the velocity $v(z)$ is a function only of depth. This is a reasonable starting point, even though it neglects the even more complicated distortions that occur when there are lateral velocity variations. We will also examine data that shows plane-like waves and spherical-like waves resulting when waves from a point source bounce back from a planar reflector.

TRAVEL-TIME DEPTH

Echo soundings give us a picture of the earth. A zero-offest section, for example, is a planar display of traces where the horizontal axis runs along the earth's surface and the vertical axis, running down, seems to measure depth, but actually measures the two-way echo delay time. Thus, in practice the vertical axis is almost never depth z; it is the *vertical travel time* τ. In a constant-velocity earth the time and the depth are related by a simple scale factor, the speed of sound. This is analogous to the way that astronomers measure distances in light-years, always referencing the speed of light. The meaning of the scale factor in seismic imaging is that the (x, τ)-plane has a vertical exaggeration compared to the (x, z)-plane. In reconnaissance work, the vertical is often exaggerated by about a factor of five. By the time prospects have been sufficiently narrowed for a drill site to be selected, the vertical exaggeration factor in use is likely to be about unity (no exaggeration).

In seismic reflection imaging, the waves go down and then up, so the **traveltime depth** τ is defined as two-way vertical travel time.

$$\tau \quad = \quad \frac{2\,z}{v} \quad . \tag{1}$$

This is the convention that I have chosen to use throughout this book.

Vertical exaggeration

The first task in interpretation of seismic data is to figure out the approximate numerical value of the **vertical exaggeration**. The vertical exaggeration is $2/v$ because it is the ratio of the apparent slope $\Delta\tau/\Delta x$ to the actual slope $\Delta z/\Delta x$ where $\Delta\tau = 2\ \Delta z/v$. Since the velocity generally *increases* with depth, the **vertical exaggeration** generally *decreases* with depth.

For velocity-stratified media, the time-to-depth conversion formula is

$$\tau(z) \quad = \quad \int_0^z \frac{2\ dz}{v(z)} \qquad \text{or} \qquad \frac{d\tau}{dz} \quad = \quad \frac{2}{v} \tag{2}$$

HORIZONTALLY MOVING WAVES

In practice, horizontally going waves are easy to recognize because their travel time is a linear function of the offset distance between shot and receiver. There are two kinds of horizontally going waves, one where the traveltime line goes through the origin, and the other where it does not. When the line goes through the origin, it means the ray path is always near the earth's surface where the sound source and the receivers are located. (Such waves are called "**ground roll**" on land or "**guided wave**s" at sea; sometimes they are just called "**direct arrivals**".)

When the traveltime line does not pass through the origin it means parts of the ray path plunge into the earth. This is usually explained by the unlikely looking rays shown in Figure 1 which frequently occur in practice. Later in this chapter we will see that

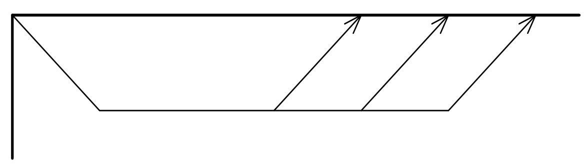

Figure 1: Rays associated with **head wave**s.

Snell's law predicts these rays in a model of the earth with two layers, where the deeper layer is faster and the ray bottom is along the interface between the slow medium and the fast medium. Meanwhile, however, notice that these ray paths imply data with a linear travel time versus distance corresponding to increasing ray length along the ray bottom. Where the ray is horizontal in the lower medium, its wavefronts are vertical. These waves are called "**head wave**s," perhaps because they are typically fast and arrive *ahead* of other waves.

Amplitudes

The nearly vertically-propagating waves (reflections) spread out essentially in three dimensions, whereas the nearly horizontally-going waves never get deep into the earth because, as we will see, they are deflected back upward by the velocity gradient. Thus horizontal waves spread out in essentially two dimensions, so that energy conservation suggests that their amplitudes should dominate the amplitudes of reflections on raw data. This is often true for **ground roll**. Head waves, on the other hand, are often much weaker, often being visible only because they often arrive before more energetic waves. The weakness of **head wave**s is explained by the small percentage of solid angle occupied by the waves leaving a source that eventually happen to match up with layer boundaries and propagate as head waves. I selected the examples below because of the strong headwaves. They are nearly as strong as the guided waves. To compensate for diminishing energy with distance, I scaled data displays by multiplying by the offset distance between the shot and the receiver.

In data display, the slowness (slope of the time-distance curve) is often called the **stepout** p. Other commonly-used names for this slope are **time dip** and **reflection slope**. The best way to view waves with **linear moveout** is after time shifting to remove a standard linear moveout such as that of water. An equation for the shifted time is

$$\tau \quad = \quad t - px \tag{3}$$

where p is often chosen to be the inverse of the velocity of water, namely, about 1.5 km/s, or $p = .66$s/km and $x = 2h$ is the horizontal separation between the sound source and receiver, usually referred to as the **offset**.

Ground roll and **guided wave**s are typically slow because materials near the earth's surface typically are slow. Slow waves are steeply sloped on a time-versus-offset display. It is not surprising that marine guided waves typically have speeds comparable to water waves (near 1.47 km/s approximately 1.5 km/s). It is perhaps surprising that **ground roll** also often has the speed of sound in water. Indeed, the depth to underground water is often determined by seismology before drilling for water. Ground roll also often has a speed comparable to the speed of sound in air, 0.3 km/sec, though, much to my annoyance I could not find a good example of it today. Figure 2 is an example of energetic **ground roll** (land) that happens to have a speed close to that of water.

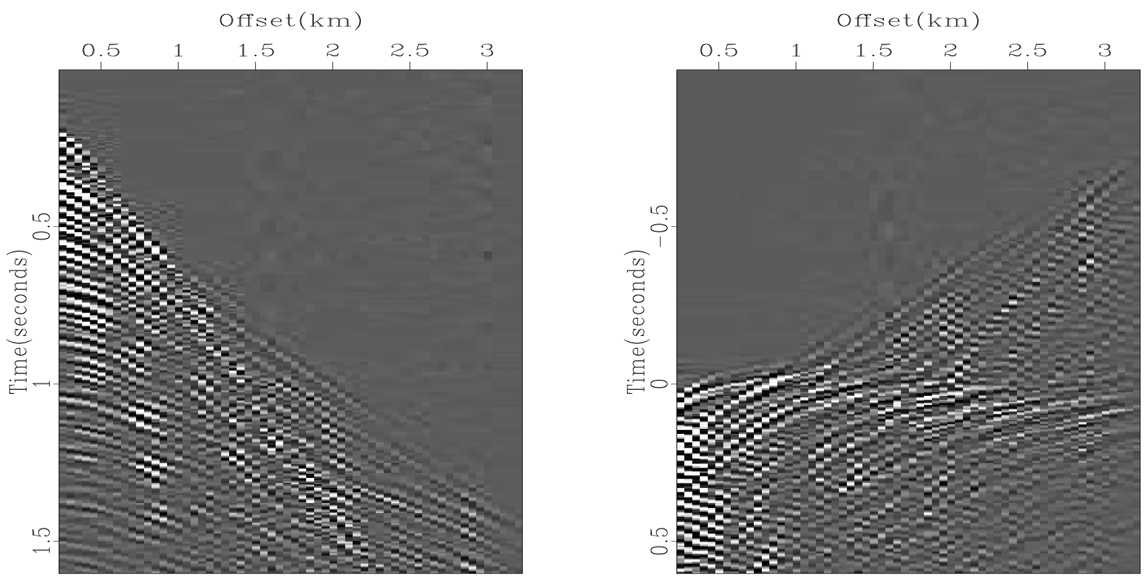

Figure 2: Land shot profile (Yilmaz and Cumro) #39 from the Middle East before (left) and after (right) linear moveout at water velocity.

The speed of a ray traveling along a layer interface is the rock speed in the faster layer (nearly always the lower layer). It is not an average of the layer above and the layer below.

Figures 3 and 4 are examples of energetic marine guided waves. In Figure 3 at $\tau = 0$ (designated `t-t_water`) at small offset is the wave that travels directly from the shot to the receivers. This wave dies out rapidly with offset (because it interferes with a wave of opposite polarity reflected from the water surface). At near offset slightly later than $\tau = 0$ is the water bottom reflection. At wide offset, the water bottom reflection is quickly followed by multiple reflections from the bottom. Critical angle reflection is defined as where the **head wave** comes tangent to the reflected wave. Before (above) $\tau = 0$ are the **head waves**. There are two obvious slopes, hence two obvious layer interfaces. Figure 4 is much like Figure 3 but the water bottom is shallower.

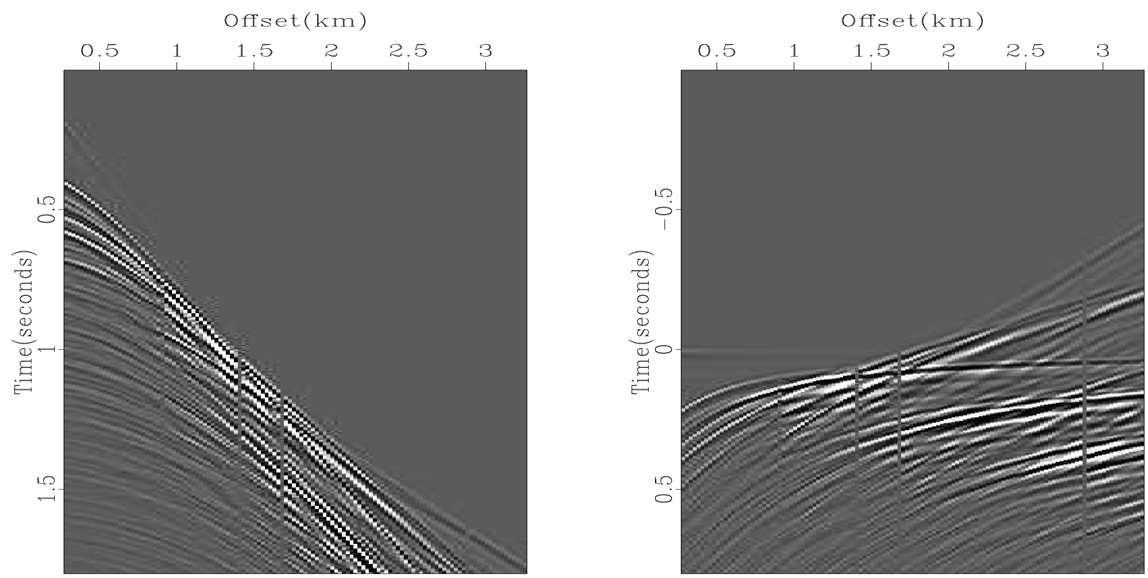

Figure 3: Marine shot profile (Yilmaz and Cumro) #20 from the Aleutian Islands.

Figure 5 shows data where the first arriving energy is not along a few straight line segments, but is along a curve. This means the velocity increases smoothly with depth as soft sediments compress.

LMO by nearest-neighbor interpolation

To do **linear moveout** (**LMO**) correction, we need to time-shift data. Shifting data requires us to interpolate it. The easiest interpolation method is the nearest-neighbor method. We begin with a signal given at times `t = t0+dt*it` where `it` is an integer. Then we can use equation (3), namely $\tau = t - px$. Given the location `tau` of the desired value we backsolve for an integer, say `itau`. In C, conversion of a real value to an integer is done by truncating the fractional part of the real value. To get rounding up as well as down, we add `0.5` before conversion to an integer, namely `itau=0.5+(tau-tau0)/dt`. This gives the nearest neighbor. The way the program works is to identify two points, one in (t, x)-space and one in (τ, x)-space. Then the data value at one point in one space is carried to the other. The adjoint operation copies τ space back to t space.

Nearest neighbor rounding is crude but ordinarily very reliable. I discovered a very rare numerical roundoff problem peculiar to signal time-shifting, a problem which arises in the linear moveout application when the water velocity, about 1.48 km/sec is approximated by 1.5=3/2. The problem arises only where the amount of the time shift is a numerical value (like 12.5000001 or 12.499999) and the fractional part should be exactly 1/2 but numerical

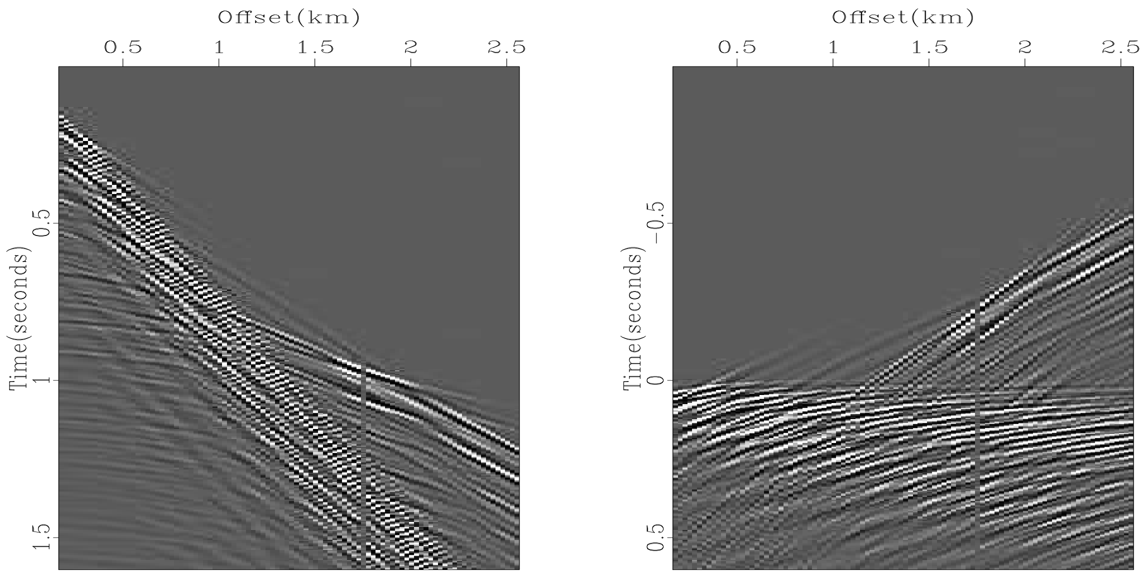

Figure 4: Marine shot profile (Yilmaz and Cumro) #32 from the North Sea.

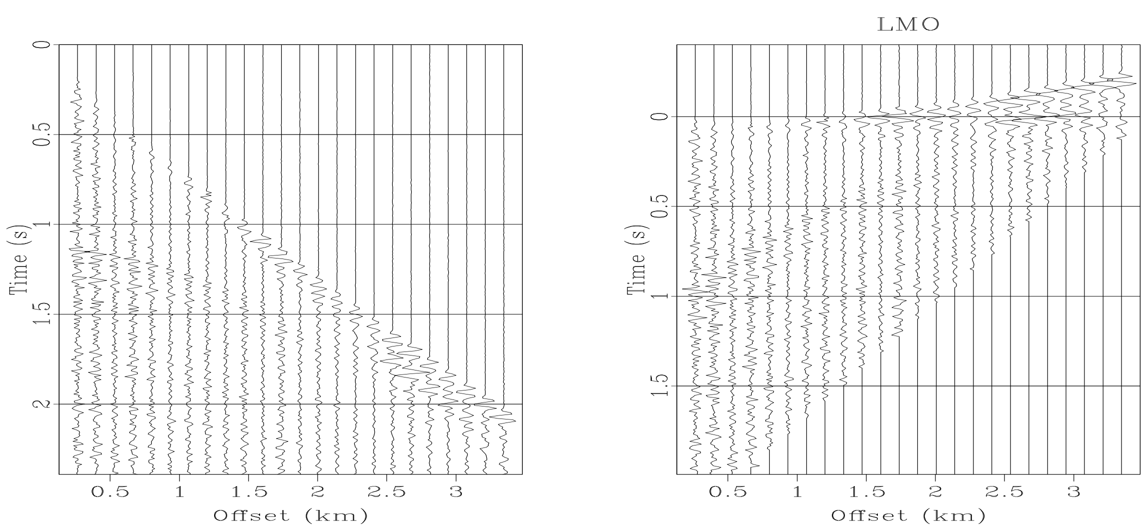

Figure 5: A common midpoint gather from the Gulf of Mexico before (left) and after (right) linear moveout at water velocity. Later I hope to estimate velocity with depth in shallow strata. Press button for **movie** over midpoint.

rounding pushes it randomly in either direction. We would not care if an entire signal was shifted by either 12 units or by 13 units. What is troublesome, however, is if some random portion of the signal shifts 12 units while the rest of it shifts 13 units. Then the output signal has places which are empty while adjacent places contain the sum of two values. Linear moveout is the only application where I have ever encountered this difficulty. The problem disappears if we use a more accurate sound velocity or if we switch from nearest-neighbor interpolation to linear interpolation.

Muting

Surface waves are a mathematician's delight because they exhibit many complex phenomena. Since these waves are often extremely strong, and since the information they contain about the earth refers only to the shallowest layers, typically, considerable effort is applied to array design in field recording to suppress these waves. Nevertheless, in many areas of the earth, these pesky waves may totally dominate the data.

A simple method to suppress **ground roll** in data processing is to multiply a strip of data by a near-zero weight (the mute). To reduce truncation artifacts, the mute should taper smoothly to zero (or some small value). Because of the extreme variability from place to place on the earth's surface, there are many different philosophies about designing mutes. Some mute programs use a data dependent weighting function (such as automatic gain control). Subroutine `mutter()` on the following page, however, operates on a simpler idea: the user supplies trajectories defining the mute zone.

Figure 6 shows an example of use of the routine `mutter()` on the next page on the shallow water data shown in Figure 5.

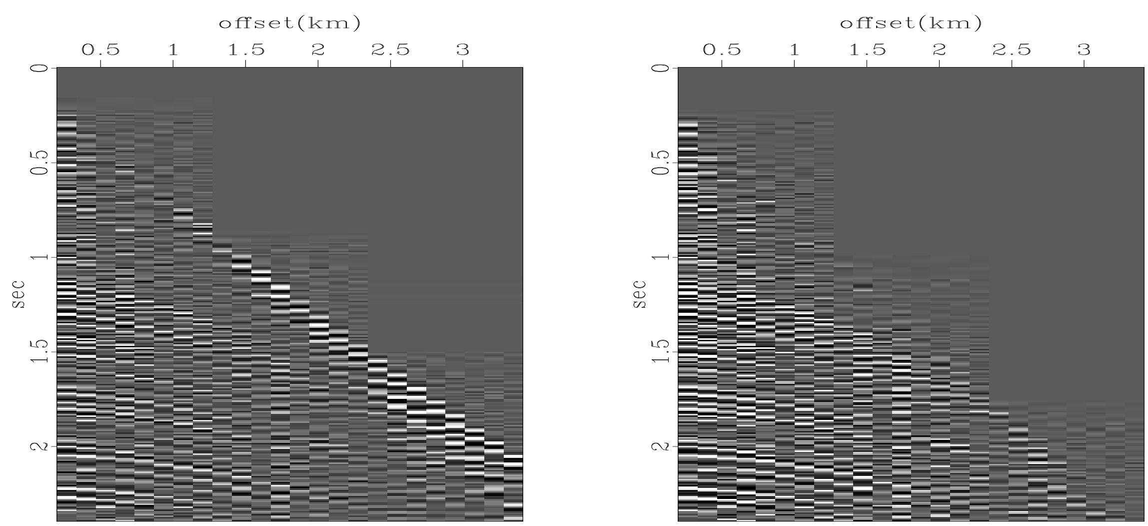

Figure 6: Jim's first gather before and after muting.

system/seismic/mutter.c

```
void mutter (float tp      /* time step */,
             float slope0  /* first slope */,
             float slopep  /* second slope */,
             float x       /* offset */,
             float *data   /* trace */)
/*< Mute >*/
{
    int it;
    float wt, t;

    if (abs0) x = fabsf(x);

    for (it=0; it < nt; it++) {
        t = t0+it*dt;
        if (hyper) t *= t;
        wt = t - x * slope0;
        if ((inner && wt > 0.) || (!inner && wt < 0.)) {
            data[it] = 0.;
        } else {
            wt = t - tp - x * slopep;
            if ((inner && wt >=0.) || (!inner && wt <= 0.)) {
                wt = sinf(0.5 * SF_PI *
                          (t-x*slope0)/(tp+x*(slopep-slope0)));
                data[it] *= (wt*wt);
            }
        }
    }
}
```

DIPPING WAVES

Above we considered waves going vertically and waves going horizontally. Now let us consider waves propagating at the intermediate angles. For the sake of definiteness, I have chosen to consider only downgoing waves in this section. We will later use the concepts developed here to handle both downgoing and upcoming waves.

Rays and fronts

It is natural to begin studies of waves with equations that describe plane waves in a medium of constant velocity.

Figure 7 depicts a ray moving down into the earth at an angle θ from the vertical. Perpendicular to the ray is a wavefront. By elementary geometry the angle between the

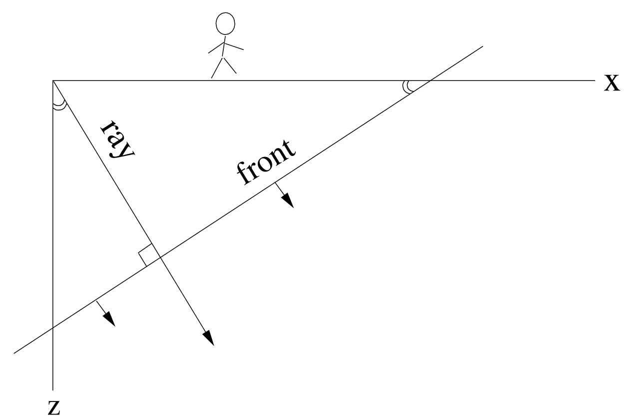

Figure 7: Downgoing ray and wavefront.

wave**front** and the earth's surface is also θ. The **ray** increases its length at a speed v. The speed that is observable on the earth's surface is the intercept of the wavefront with the earth's surface. This speed, namely $v/\sin\theta$, is faster than v. Likewise, the speed of the intercept of the wavefront and the vertical axis is $v/\cos\theta$. A mathematical expression for a straight line like that shown to be the wavefront in Figure 7 is

$$z \quad = \quad z_0 \ - \ x \ \tan\theta \tag{4}$$

In this expression z_0 is the intercept between the wavefront and the vertical axis. To make the intercept move downward, replace it by the appropriate velocity times time:

$$z \quad = \quad \frac{v\,t}{\cos\theta} \ - \ x \ \tan\theta \tag{5}$$

Solving for time gives

$$t(x,z) \quad = \quad \frac{z}{v}\ \cos\theta \ + \ \frac{x}{v}\ \sin\theta \tag{6}$$

Equation (6) tells the time that the wavefront will pass any particular location (x,z). The expression for a shifted waveform of arbitrary shape is $f(t-t_0)$. Using (6) to define the time shift t_0 gives an expression for a wavefield that is some waveform moving on a **ray**.

$$\text{moving wavefield} \quad = \quad f\left(t \ - \ \frac{x}{v}\ \sin\theta \ - \ \frac{z}{v}\ \cos\theta \right) \tag{7}$$

Snell waves

In reflection seismic surveys the velocity contrast between shallowest and deepest reflectors ordinarily exceeds a factor of two. Thus depth variation of velocity is almost always included in the analysis of field data. Seismological theory needs to consider waves that are just like plane waves except that they bend to accommodate the velocity stratification $v(z)$. Figure 8 shows this in an idealized geometry: waves radiated from the horizontal flight of a supersonic airplane. The airplane passes location x at time $t_0(x)$ flying horizontally at a

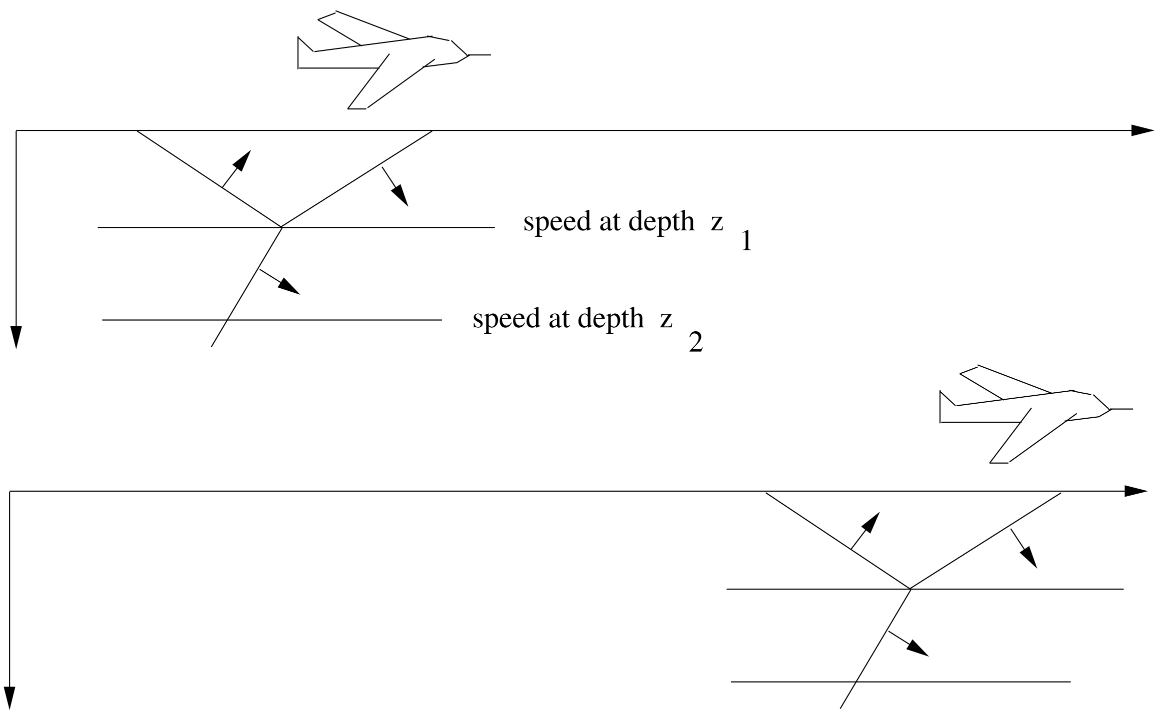

Figure 8: Fast airplane radiating a sound wave into the earth. From the figure you can deduce that the horizontal speed of the wavefront is the same at depth z_1 as it is at depth z_2. This leads (in isotropic media) to Snell's law.

constant speed. Imagine an earth of horizontal plane layers. In this model there is nothing to distinguish any point on the x-axis from any other point on the x-axis. But the seismic velocity varies from layer to layer. There may be reflections, head waves, shear waves, converted waves, anisotropy, and multiple reflections. Whatever the picture is, it moves along with the airplane. A picture of the wavefronts near the airplane moves along with the airplane. The top of the picture and the bottom of the picture both move laterally at the same speed even if the earth velocity increases with depth. If the top and bottom didn't go at the same speed, the picture would become distorted, contradicting the presumed symmetry of translation. This horizontal speed, or rather its inverse $\partial t_0/\partial x$, has several names. In practical work it is called the ***stepout***. In theoretical work it is called the ***ray parameter***. It is very important to note that $\partial t_0/\partial x$ does not change with depth, even though the seismic velocity does change with depth. In a constant-velocity medium, the angle of a wave does not change with depth. In a stratified medium, $\partial t_0/\partial x$ does not change with depth.

Figure 9 illustrates the differential geometry of the wave. Notice that triangles have their hypotenuse on the x-axis and the z-axis but not along the ray. That's because this figure refers to wave fronts. (If you were thinking the hypotenuse would measure $v\Delta t$, it could be you were thinking of the tip of a ray and its projection onto the x and z axes.) The diagram shows that

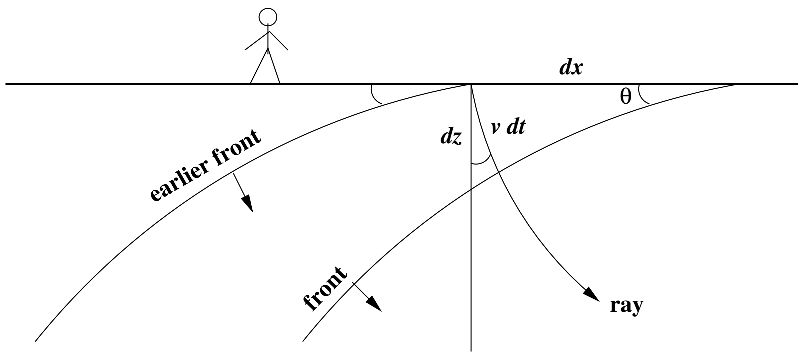

Figure 9: Downgoing fronts and rays in stratified medium $v(z)$. The wavefronts are horizontal translations of one another.

$$\frac{\partial t_0}{\partial x} = \frac{\sin\theta}{v} \tag{8}$$

$$\frac{\partial t_0}{\partial z} = \frac{\cos\theta}{v} \tag{9}$$

These two equations define two (inverse) speeds. The first is a horizontal speed, measured along the earth's surface, called the *horizontal* ***phase velocity.*** The second is a vertical speed, measurable in a borehole, called the *vertical phase velocity.* Notice that both these speeds *exceed* the velocity v of wave propagation in the medium. Projection of wave *fronts* onto coordinate axes gives speeds larger than v, whereas projection of *rays* onto coordinate axes gives speeds smaller than v. The inverse of the phase velocities is called the ***stepout*** or the ***slowness***.

Snell's law relates the angle of a wave in one layer with the angle in another. The constancy of equation (8) in depth is really just the statement of Snell's law. Indeed, we have just derived Snell's law. All waves in seismology propagate in a velocity-stratified medium. So they cannot be called plane waves. But we need a name for waves that are near to plane waves. A ***Snell wave*** will be defined to be the generalization of a plane wave to a stratified medium $v(z)$. A plane wave that happens to enter a medium of depth-variable velocity $v(z)$ gets changed into a Snell wave. While a plane wave has an angle of propagation, a Snell wave has instead a ***Snell parameter*** $p = \partial t_0/\partial x$.

It is noteworthy that Snell's parameter $p = \partial t_0/\partial x$ is directly observable at the surface, whereas neither v nor θ is directly observable. Since $p = \partial t_0/\partial x$ is not only observable, but constant in depth, it is customary to use it to eliminate θ from equations (8) and (9):

$$\frac{\partial t_0}{\partial x} = \frac{\sin\theta}{v} = p \tag{10}$$

$$\frac{\partial t_0}{\partial z} = \frac{\cos\theta}{v} = \sqrt{\frac{1}{v(z)^2} - p^2} \tag{11}$$

Evanescent waves

Suppose the velocity increases to infinity at infinite depth. Then equation (11) tells us that something strange happens when we reach the depth for which p^2 equals $1/v(z)^2$. That is the depth at which the ray turns horizontal. We will see in a later chapter that below this critical depth the seismic wavefield damps exponentially with increasing depth. Such waves are called **evanescent**. For a physical example of an evanescent wave, forget the airplane and think about a moving bicycle. For a bicyclist, the slowness p is so large that it dominates $1/v(z)^2$ for all earth materials. The bicyclist does not radiate a wave, but produces a ground deformation that decreases exponentially into the earth. To radiate a wave, a source must move faster than the material velocity.

Solution to kinematic equations

The above differential equations will often reoccur in later analysis, so they are very important. Interestingly, these differential equations have a simple solution. Taking the Snell wave to go through the origin at time zero, an expression for the arrival time of the Snell wave at any other location is given by

$$t_0(x,z) \quad = \quad \frac{\sin\theta}{v}\, x \ + \ \int_0^z \frac{\cos\theta}{v}\, dz \tag{12}$$

$$t_0(x,z) \quad = \quad p\, x \ + \ \int_0^z \sqrt{\frac{1}{v(z)^2} \ - \ p^2} \ dz \tag{13}$$

The validity of equations (12) and (13) is readily checked by computing $\partial t_0/\partial x$ and $\partial t_0/\partial z$, then comparing with (10) and (11).

An arbitrary waveform $f(t)$ may be carried by the Snell wave. Use (12) and (13) to *define* the time t_0 for a delayed wave $f[t - t_0(x,z)]$ at the location (x,z).

$$\text{SnellWave}(t,x,z) \quad = \quad f\left(t \ - \ p\, x \ - \ \int_0^z \sqrt{\frac{1}{v(z)^2} \ - \ p^2} \ dz \right) \tag{14}$$

Equation (14) carries an arbitrary signal throughout the whole medium. Interestingly, it does not agree with wave propagation theory or real life because equation (14) does not correctly account for amplitude changes that result from velocity changes and reflections. Thus it is said that Equation (14) is "kinematically" correct but "dynamically" incorrect. It happens that most industrial data processing only requires things to be kinematically correct, so this expression is a usable one.

CURVED WAVEFRONTS

The simplest waves are expanding circles. An equation for a circle expanding with velocity v is

$$v^2\, t^2 \quad = \quad x^2 \ + \ z^2 \tag{15}$$

Considering t to be a constant, i.e. taking a snapshot, equation (15) is that of a circle. Considering z to be a constant, it is an equation in the (x,t)-plane for a hyperbola. Considered in the (t, x, z)-volume, equation (15) is that of a cone. Slices at various values of t show circles of various sizes. Slices of various values of z show various hyperbolas.

Converting equation (15) to traveltime depth τ we get

$$v^2 t^2 \quad = \quad z^2 \; + \; x^2 \tag{16}$$

$$t^2 \quad = \quad \tau^2 \; + \; \frac{x^2}{v^2} \tag{17}$$

The earth's velocity typically increases by more than a factor of two between the earth's surface, and reflectors of interest. Thus we might expect that equation (17) would have little practical use. Luckily, this simple equation will solve many problems for us if we know how to interpret the velocity as an average velocity.

Root-mean-square velocity

When a ray travels in a depth-stratified medium, Snell's parameter $p = v^{-1} \sin\theta$ is constant along the ray. If the ray emerges at the surface, we can measure the distance x that it has traveled, the time t it took, and its apparent speed $dx/dt = 1/p$. A well-known estimate $\hat{v}$ for the earth velocity contains this apparent speed.

$$\hat{v} \quad = \quad \sqrt{\frac{x}{t}\,\frac{dx}{dt}} \tag{18}$$

To see where this velocity estimate comes from, first notice that the stratified velocity $v(z)$ can be parameterized as a function of time and take-off angle of a ray from the surface.

$$v(z) \quad = \quad v(x, z) \quad = \quad v'(p, t) \tag{19}$$

The x coordinate of the tip of a ray with Snell parameter p is the horizontal component of velocity integrated over time.

$$x(p, t) \quad = \quad \int_0^t \; v'(p, t) \;\; \sin\theta(p, t) \;\, dt \quad = \quad p \, \int_0^t v'(p, t)^2 \;\, dt \tag{20}$$

Inserting this into equation (18) and canceling $p = dt/dx$ we have

$$\hat{v} \quad = \quad v_{\rm RMS} \quad = \quad \sqrt{\frac{1}{t} \, \int_0^t v'(p, t)^2 \;\, dt} \tag{21}$$

which shows that the observed velocity is the "root-mean-square" velocity.

When velocity varies with depth, the traveltime curve is only roughly a hyperbola. If we break the event into many short line segments where the i-th segment has a slope p_i and a midpoint (t_i, x_i) each segment gives a different $\hat{v}(p_i, t_i)$ and we have the unwelcome chore of assembling the best model. Instead, we can fit the observational data to the best fitting hyperbola using a different velocity hyperbola for each apex, in other words, find $V(\tau)$ so this equation will best flatten the data in (τ, x)-space.

$$t^2 \quad = \quad \tau^2 + x^2/V(\tau)^2 \tag{22}$$

Differentiate with respect to x at constant τ getting

$$2t \, dt/dx \quad = \quad 2x/V(\tau)^2 \tag{23}$$

which confirms that the observed velocity $\hat{v}$ in equation (18), is the same as $V(\tau)$ no matter where you measure $\hat{v}$ on a hyperbola.

Layered media

From the assumption that experimental data can be fit to hyperbolas (each with a different velocity and each with a different apex τ) let us next see how we can fit an earth model of layers, each with a constant velocity. Consider the horizontal reflector overlain by a stratified **interval velocity** $v(z)$ shown in Figure 10.

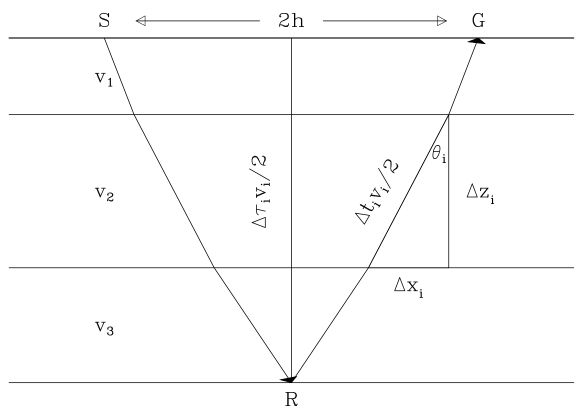

Figure 10: Raypath diagram for normal moveout in a stratified earth.

The separation between the source and geophone, also called the offset, is $2h$ and the total travel time is t. Travel times are not be precisely hyperbolic, but it is common practice to find the best fitting hyperbolas, thus finding the function $V^2(\tau)$.

$$t^2 \quad = \quad \tau^2 + \frac{4h^2}{V^2(\tau)} \tag{24}$$

where τ is the zero-offset two-way traveltime.

An example of using equation (24) to stretch t into τ is shown in Figure 11. (The programs that find the required $V(\tau)$ and do the stretching are coming up in chapter **??**.)

Equation (21) shows that $V(\tau)$ is the "root-mean-square" or "RMS" velocity defined by an average of v^2 over the layers. Expressing it for a small number of layers we get

$$V^2(\tau) \quad = \quad \frac{1}{\tau} \sum_i v_i^2 \Delta\tau_i \tag{25}$$

where the zero-offset traveltime τ is a sum over the layers:

$$\tau \quad = \quad \sum_i \Delta\tau_i \tag{26}$$

The two-way vertical travel time τ_i in the ith layer is related to the thickness Δz_i and the velocity v_i by

$$\Delta\tau_i \quad = \quad \frac{2\ \Delta z_i}{v_i} \quad . \tag{27}$$

Next we examine an important practical calculation, getting interval velocities from measured RMS velocities: Define in layer i, the interval velocity v_i and the two-way vertical travel time $\Delta\tau_i$. Define the RMS velocity of a reflection from the bottom of the i-th layer

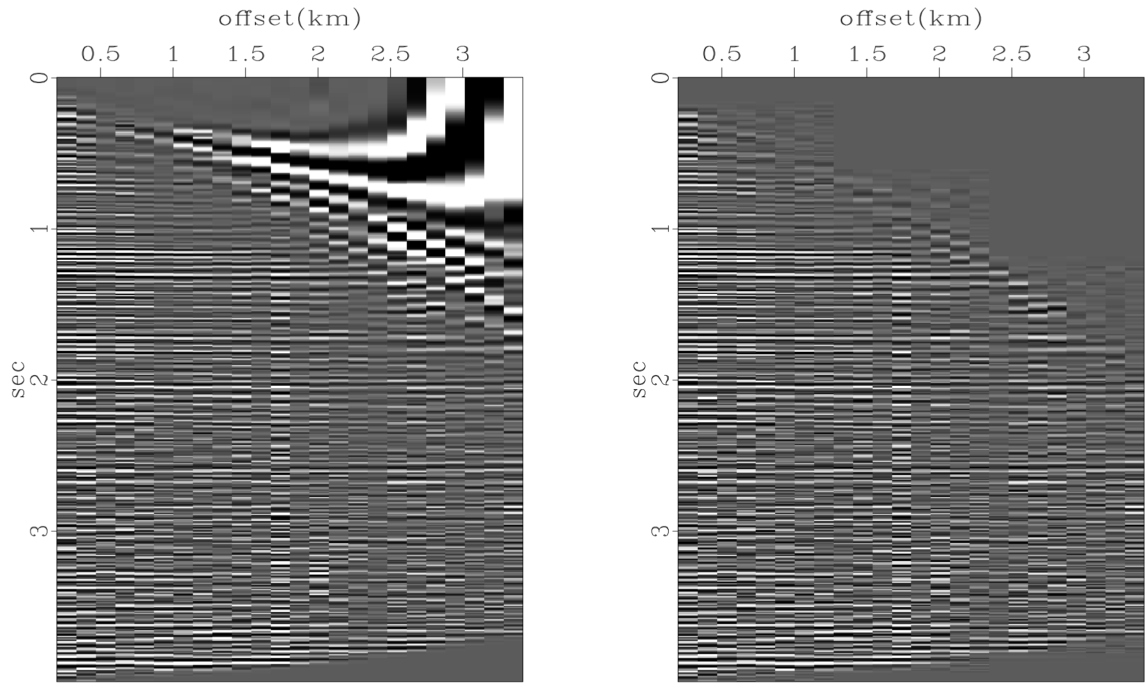

Figure 11: If you are lucky and get a good velocity, when you do NMO, everything turns out flat. Shown with and without mute.

to be V_i. Equation (25) tells us that for reflections from the bottom of the first, second, and third layers we have

$$V_1^2 = \frac{v_1^2 \Delta\tau_1}{\Delta\tau_1} \tag{28}$$

$$V_2^2 = \frac{v_1^2 \Delta\tau_1 + v_2^2 \Delta\tau_2}{\Delta\tau_1 + \Delta\tau_2} \tag{29}$$

$$V_3^2 = \frac{v_1^2 \Delta\tau_1 + v_2^2 \Delta\tau_2 + v_3^2 \Delta\tau_3}{\Delta\tau_1 + \Delta\tau_2 + \Delta\tau_3} \tag{30}$$

Normally it is easy to measure the times of the three hyperbola tops, $\Delta\tau_1$, $\Delta\tau_1 + \Delta\tau_2$ and $\Delta\tau_1 + \Delta\tau_2 + \Delta\tau_3$. Using methods in chapter **??** we can measure the RMS velocities V_2 and V_3. With these we can solve for the interval velocity v_3 in the third layer. Rearrange (30) and (29) to get

$$(\Delta\tau_1 + \Delta\tau_2 + \Delta\tau_3) V_3^2 = v_1^2 \Delta\tau_1 + v_2^2 \Delta\tau_2 + v_3^2 \Delta\tau_3 \tag{31}$$

$$(\Delta\tau_1 + \Delta\tau_2) V_2^2 = v_1^2 \Delta\tau_1 + v_2^2 \Delta\tau_2 \tag{32}$$

and subtract getting the squared interval velocity v_3^2

$$v_3^2 = \frac{(\Delta\tau_1 + \Delta\tau_2 + \Delta\tau_3) V_3^2 - (\Delta\tau_1 + \Delta\tau_2) V_2^2}{\Delta\tau_3} \tag{33}$$

For any real earth model we would not like an imaginary velocity which is what could happen if the squared velocity in (33) happened to be negative. You see that this means that the RMS velocity we estimate for the third layer cannot be too much smaller than the one we estimate for the second layer.

Nonhyperbolic curves

Occasionally data does not fit a hyperbolic curve very well. Two other simple fitting functions are

$$t^2 \quad = \quad \tau^2 \; + \; \frac{x^2}{v^2} \; + \; x^4 \times \text{parameter} \tag{34}$$

$$(t - t_0)^2 \quad = \quad (\tau - t_0)^2 \; + \; \frac{x^2}{v^2} \tag{35}$$

Equation (34) has an extra adjustable parameter of no simple interpretation other than the beginning of a power series in x^2. I prefer Equation (35) where the extra adjustable parameter is a time shift t_0 which has a simple interpretation, namely, a time shift such as would result from a near-surface low velocity layer. In other words, a datum correction.

Velocity increasing linearly with depth

Theoreticians are delighted by velocity increasing linearly with depth because it happens that many equations work out in closed form. For example, rays travel in circles. We will need convenient expressions for velocity as a function of traveltime depth and RMS velocity as a function of traveltime depth. Let us get them. We take the **interval velocity** $v(z)$ increasing linearly with depth:

$$v(z) \quad = \quad v_0 + \alpha z \tag{36}$$

This presumption can also be written as a differential equation:

$$\frac{dv}{dz} \quad = \quad \alpha. \tag{37}$$

The relationship between z and vertical two-way traveltime $\tau(z)$ (see equation (27)) is also given by a differential equation:

$$\frac{d\tau}{dz} \quad = \quad \frac{2}{v(z)}. \tag{38}$$

Letting $v(\tau) = v(z(\tau))$ and applying the chain rule gives the differential equation for $v(\tau)$:

$$\frac{dv}{dz}\frac{dz}{d\tau} \quad = \quad \frac{dv}{d\tau} \quad = \quad \frac{v\alpha}{2}, \tag{39}$$

whose solution gives us the desired expression for **interval velocity** as a function of traveltime depth.

$$v(\tau) \quad = \quad v_0 \; e^{\alpha\tau/2}. \tag{40}$$

Prior RMS velocity

Substituting the theoretical interval velocity $v(\tau)$ from equation (40) into the definition of RMS velocity $V(\tau)$ (equation (25)) yields:

$$\tau\, V^2(\tau) = \int_0^\tau v^2(\tau')\, d\tau' \tag{41}$$

$$= v_0^2\, \frac{e^{\alpha\tau} - 1}{\alpha}. \tag{42}$$

Thus the desired expression for RMS velocity as a function of traveltime depth is:

$$V(\tau) = v_0 \sqrt{\frac{e^{\alpha\tau} - 1}{\alpha\tau}} \tag{43}$$

For small values of $\alpha\tau$, this can be approximated as

$$V(\tau) \approx v_0 \sqrt{1 + \alpha\tau/2}. \tag{44}$$