Generalized nonhyperbolic moveout approximation[a]

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ABSTRACT

Reflection moveout approximations are commonly used for velocity analysis, stacking, and time migration. We introduce a novel functional form for approximating the moveout of reflection traveltimes at large offsets. While the classic hyperbolic approximation uses only two parameters (the zero-offset time and the moveout velocity), our form involves five parameters, which can be determined, in a known medium, from zero-offset computations and from tracing one non-zero-offset ray. We call it a generalized approximation because it reduces to some known three-parameter forms (the shifted hyperbola of Malovichko, de Baziliere, and Castle; the Padé approximation of Alkhalifah and Tsvankin; and others) with a particular choice of coefficients. By testing the accuracy of the proposed approximation with analytical and numerical examples, we show that it can bring several-orders-of-magnitude improvement in accuracy at large offsets compared to known analytical approximations, which makes it as good as exact for many practical purposes.

INTRODUCTION

Reflection moveout approximations are commonly used for velocity analysis, stacking, and time migration (Yilmaz, 2000). The reflection traveltime as a function of the source-receiver offset has a well-known hyperbolic form in the case of plane reflectors in homogeneous isotropic (or elliptically anisotropic) overburden. A hyperbolic behavior of the PP moveout is always valid around the zero offset thanks to the source-receiver reciprocity and the first-order Taylor series expansion. However, any deviations from this simple model may cause nonhyperbolic behavior at large offsets (Fomel and Grechka, 2001).

Considerable research effort has been devoted to developing nonhyperbolic moveout approximations in both isotropic and anisotropic media. The work on isotropic approximations goes back to Bolshykh (1956), Taner and Koehler (1969), Malovichko (1978), de Bazelaire (1988), Castle (1994), and others. Fowler (2003) provides a comprehensive review of many different approximations developed for non-hyperbolic moveout in anisotropic (VTI – vertically transversally isotropic) media. A particularly simple "velocity acceleration" model for nonhyperbolic moveout is suggested by

Taner et al. (2005, 2007). Causse (2004) approximates nonhyperbolic moveout by expanding it into a sum of basis functions. Douma and Calvert (2006) and Douma and van der Baan (2008) build an accurate moveout approximation by using rational interpolation between several rays.

In this paper, we propose a general functional form for nonhyperbolic approximations that can be applied to different kinds of seismic media. The proposed form includes five coefficients as opposed to two coefficients in the classic hyperbolic approximation. In certain cases, the number of coefficients can be reduced. In the case of a homogeneous VTI medium and the "acoustic approximation" of Alkhalifah (1998), our approximation becomes identical to the one proposed previously by Fomel (2004). In the general case, determining the optimal coefficients requires tracing of only one non-zero-offset ray.

Using analytical ray-tracing solutions and numerical experiments, we compare the accuracy of our approximation with the accuracy of other known approximations and discover an improvement in accuracy of several orders of magnitude. Potential applications of the new approximation include velocity analysis and time-domain imaging.

NONHYPERBOLIC MOVEOUT APPROXIMATION

Let $t(x)$ represent the reflection traveltime as a function of the source-receiver offset x. We propose the following general form of the moveout approximation:

$$t^2(x) \approx (1-\xi)\,(t_0^2 + a\,x^2) + \xi\,\sqrt{t_0^4 + 2\,b\,t_0^2\,x^2 + c\,x^4}\;. \tag{1}$$

The five parameters a, b, c, ξ, and t_0 describe the moveout behavior. By simple algebraic manipulations, one can also rewrite equation 1 as

$$t^2(x) \approx t_0^2 + \frac{x^2}{v^2} + \frac{A\,x^4}{v^4\left(t_0^2 + B\,\frac{x^2}{v^2} + \sqrt{t_0^4 + 2\,B\,t_0^2\,\frac{x^2}{v^2} + C\,\frac{x^4}{v^4}}\right)}\;, \tag{2}$$

where the new set of parameters A, B, C, v, and t_0 is related to the previous set by the equalities

$$a = \frac{A\,B + B^2 - C}{v^2\,(A + B^2 - C)}\;; \tag{3}$$

$$b = \frac{B}{v^2}\;; \tag{4}$$

$$c = \frac{C}{v^4}\;; \tag{5}$$

$$\xi = \frac{A}{C - B^2}\;. \tag{6}$$

The inverse transform is given by

$$v^2 = \frac{1}{a\,(1-\xi)+b\,\xi}\;; \tag{7}$$

$$A = \frac{\xi\,(c-b^2)}{[a\,(1-\xi)+b\,\xi]^2}\;; \tag{8}$$

$$B = \frac{b}{a\,(1-\xi)+b\,\xi}\;; \tag{9}$$

$$C = \frac{c}{[a\,(1-\xi)+b\,\xi]^2}\;. \tag{10}$$

The existence of the nonhyperbolic part in the traveltime approximation 1 and 2 is controlled by parameter A. When A is zero (which implies that $\xi = 0$ or $c = b^2$), approximation 1 is hyperbolic. When both B and C are very large, approximation 2 also reduces to the hyperbolic form.

Connection with other approximations

Equations 1-2 reduce to some well-known approximations with special choices of parameters.

- If $A = 0$, the proposed approximation reduces to the classic hyperbolic form

$$t^2(x) \approx t_0^2 + \frac{x^2}{v^2}\;, \tag{11}$$

  which is a two-parameter approximation.

- The choice of parameters $A = (1-s)/2$; $B = s/2$; $C = 0$ reduces the proposed approximation to the shifted hyperbola (Malovichko, 1978; de Bazelaire, 1988; Castle, 1994), which is the following three-parameter approximation:

$$t(x) \approx t_0\left(1-\frac{1}{s}\right) + \frac{1}{s}\sqrt{t_0^2 + s\,\frac{x^2}{v^2}}\;. \tag{12}$$

- The choice of parameters $A = -4\,\eta$; $B = 1+2\,\eta$; $C = (1+2\,\eta)^2$ reduces approximation 2 to the form proposed by Alkhalifah and Tsvankin (1995) for VTI media, which is the following three-parameter approximation:

$$t^2(x) \approx t_0^2 + \frac{x^2}{v^2} - \frac{2\,\eta\,x^4}{v^4\left[t_0^2 + (1+2\,\eta)\,\dfrac{x^2}{v^2}\right]}\;. \tag{13}$$

- The choice of parameters $A = -2\,\gamma\,t_0^2\,v^2$; $B = -A/2$; $C = A^2/4$ reduces approximation 2 to the following three-parameter approximation suggested by

Blias (2007) and reminiscent of the “velocity acceleration” equation proposed by Taner et al. (2005, 2007):

$$t^2(x) \approx t_0^2 + \frac{x^2}{v^2\,(1+\gamma\,x^2)}\;. \tag{14}$$

- The choice of parameters $A=(1-s)/2$; $B=1$; $C=2-s$ reduces the proposed approximation to the following three-parameter approximation suggested by Blias (2009):

$$t(x) \approx \frac{1}{2}\sqrt{t_0^2 + \left(1-\sqrt{s-1}\right)\,\frac{x^2}{v^2}} + \frac{1}{2}\sqrt{t_0^2 + \left(1+\sqrt{s-1}\right)\,\frac{x^2}{v^2}}\;. \tag{15}$$

- The choice of parameters $B=0$; $C=2\,A$ reduces the proposed approximation to the following three-parameter approximation also suggested by Blias (2009):

$$t^2(x) \approx \frac{t_0^2}{2} + \frac{x^2}{v^2} + \frac{1}{2}\sqrt{t_0^4 + \frac{2\,A\,x^4}{v^4}}\;. \tag{16}$$

- The choice of parameters $A=2\,\tan^2\theta$, $B=1-\tan^2\theta$, $C=1/\cos^4\theta$ reduces the proposed approximation to the double-square-root expression

$$\begin{aligned} t(x) &\approx \frac{1}{2}\sqrt{t_0^2 + \frac{x\,(x+t_0\,v\,\sin 2\theta)}{v^2\,\cos^2\theta}} + \frac{1}{2}\sqrt{t_0^2 + \frac{x\,(x-t_0\,v\,\sin 2\theta)}{v^2\,\cos^2\theta}} \\ &= \frac{\sqrt{z^2+(y+x/2)^2}}{V} + \frac{\sqrt{z^2+(y-x/2)^2}}{V}\;, \end{aligned} \tag{17}$$

  where $V=v\,\cos\theta$, $z=(t_0\,V/2)\,\cos\theta$, and $y=(t_0\,V/2)\,\sin\theta$. Equation 17 describes moveout precisely for the case of a diffraction point in a constant velocity medium.

Thus, the proposed approximation encompasses some other known forms but introduces more degrees of freedom for optimal fitting.

General method for parameter selection

Zero-offset ray

The Taylor expansion of approximation 2 around the zero offset

$$t^2(x) = t_0^2 + \frac{x^2}{v^2} + \frac{A}{2}\,\frac{x^4}{v^4\,t_0^2} + O(x^6) \tag{18}$$

provides a convenient method for evaluating coefficients t_0, v, and A by matching expansion 18 to the corresponding expansion of the exact traveltime. This is the method used previously for deriving approximations 11 and 12.

In the special case of an isotropic $V(z)$ medium, the coefficients are readily available and reduce to statistical averages of the velocity distribution (Bolshykh, 1956)

$$t_0 = 2\,m_{-1}\;, \tag{19}$$

$$v^2 = \frac{m_1}{m_{-1}}\;, \tag{20}$$

$$A = \frac{1}{2}\left(1-\frac{m_3\,m_{-1}}{m_1^2}\right)\;, \tag{21}$$

where

$$m_k = \int\limits_0^z V^k(\zeta)\,d\zeta$$

Equations 19-21 are easily extensible to the vertical transverse isotropy (VTI) case (Lyakhovitsky and Nevskiy, 1971; Blias, 1983; Alkhalifah, 1997b; Ursin and Stovas, 2006).

Nonzero-offset ray

To determine uniquely the remaining coefficients B and C, we propose to use just one additional ray reflected at a nonzero offset. Suppose that a reflection ray with the ray parameter P arrives at the offset X and traveltime T. Substituting approximation 2 into equations $t(X) = T$ and $dt/dX = P$ and solving for B and C produces the explicit analytical solution

$$B = \frac{t_0^2\,(X - P\,T\,v^2)}{X\,(t_0^2 - T^2 + P\,T\,X)} - \frac{A\,X^2}{X^2 + v^2\,(t_0^2 - T^2)}\;, \tag{22}$$

$$C = \frac{t_0^4\,(X - P\,T\,v^2)^2}{X^2\,(t_0^2 - T^2 + P\,T\,X)^2} + \frac{2\,A\,v^2\,t_0^2}{X^2 + v^2\,(t_0^2 - T^2)}\;. \tag{23}$$

Horizontal ray

If the reference ray happens to be horizontal, both X and T are infinite, and equations 22-23 are not directly applicable. However, one can use the same principle and match two terms for the behavior of the traveltime at infinitely large offsets. If the traveltime behaves as

$$t^2(x) \approx T_\infty^2 + P_\infty^2\,x^2 \tag{24}$$

for x approaching infinity, then, matching the corresponding behavior of approximation 2, we find that

$$B = \frac{t_0^2\,(1 - v^2\,P_\infty^2)}{t_0^2 - T_\infty^2} - \frac{A}{1 - v^2\,P_\infty^2}\;, \tag{25}$$

$$C = \frac{t_0^4\,(1 - v^2\,P_\infty^2)^2}{(t_0^2 - T_\infty^2)^2}\;. \tag{26}$$

ACCURACY TESTS

To illustrate the applicability of the proposed approximation, we try several analytical and numerical models. Using these models, we test the proposed approximation against the hyperbolic approximation 11, the shifted hyperbola approximation 12, and the Alkhalifah-Tsvankin approximation 13.

Analytical examples

Linear velocity and linear sloth

We start with two analytical isotropic three-parameter models: linear velocity model (described in Appendix A) and linear sloth model (described in Appendix B). In both models, it is possible to compute the exact moveout analytically and thus to compare directly the accuracy of different approximations with the exact moveout. We show this comparison in Figures 1 and 2, where the relative absolute approximation error is plotted for different approximations against a large range of the offset-to-depth ratio and the maximum-to-minimum velocity ratio. As evident from the figures, three-parameter approximations (shifted-hyperbola and Alkhalifah-Tsvankin) improve the accuracy of the two-parameter hyperbolic approximation. However, the proposed five-parameter generalized approximation brings a more significant improvement and reduces the error by several orders of magnitude.

Curved reflector in a constant-velocity medium

Our next analytical example is a curved reflector under a constant-velocity overburden. The reflector curvature is one of the possible causes of non-hyperbolic moveout (Fomel and Grechka, 2001). The Taylor expansion around zero offset for the case of a curved reflector has the form of equation 18 with the following set of parameters (Fomel, 1994)

$$t_0 = \frac{2\,L}{V}\;, \tag{27}$$

$$v = \frac{V}{\cos\beta}\;, \tag{28}$$

$$A = 2\,\tan^2\beta\,G\;, \tag{29}$$

where L is the length of the normal (zero-offset) ray, V is the true velocity, β is the reflector dip angle at the normal reflection point, $G = K\,L/(1 + K\,L)$, and K is the reflector curvature at the normal reflection point.

The two additional parameters B and C depend on the particular shape of the reflector. In the case of a hyperbolic reflector, analyzed in Appendix C, equation 2

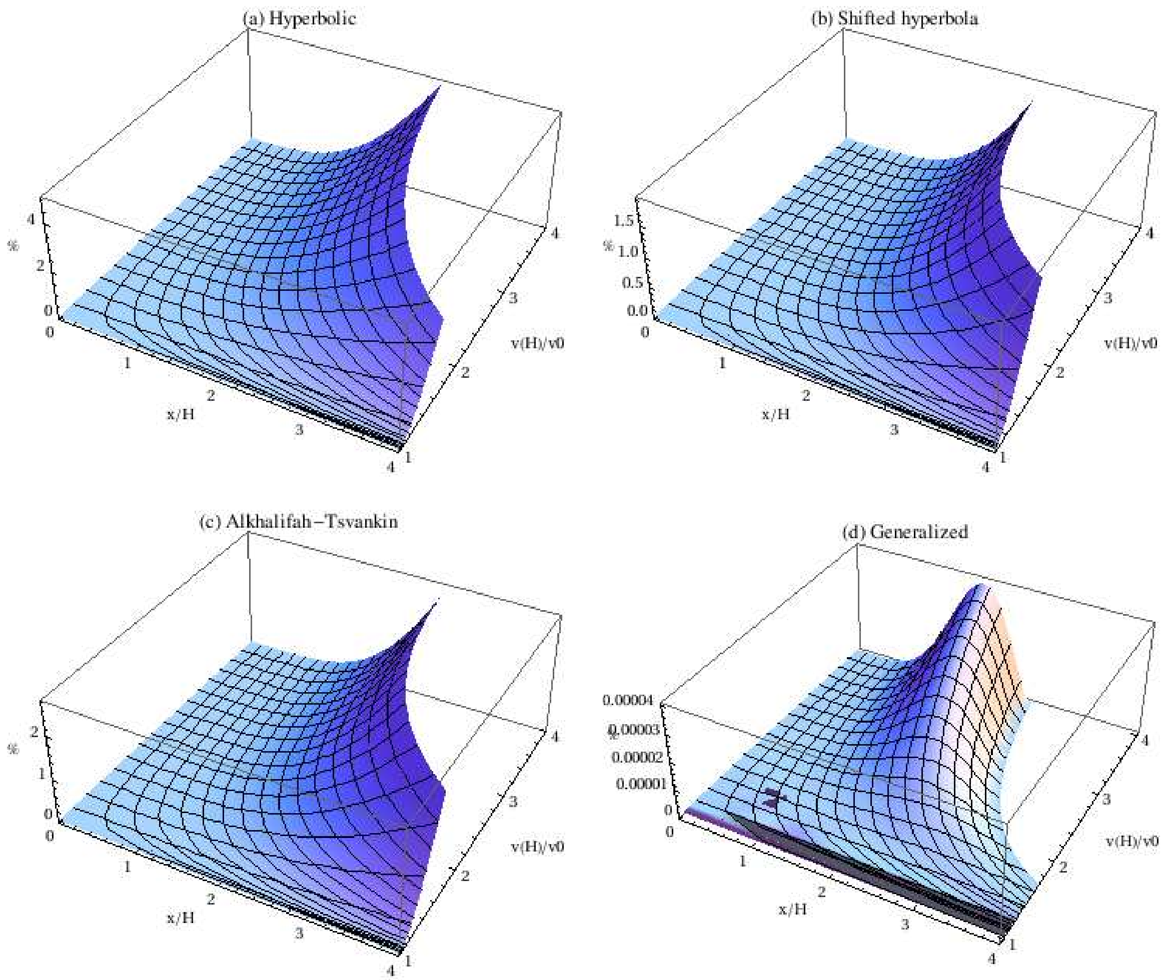

Figure 1: Relative absolute error of different traveltime approximations as a function of velocity contrast and offset/depth ratio for the case of a linear velocity model. (a) Hyperbolic approximation, (b) Shifted hyperbola approximation, (c) Alkhalifah-Tsvankin approximation, (d) Generalized nonhyperbolic approximation. The proposed generalized approximation reduces the maximum approximation error by several orders of magnitude.

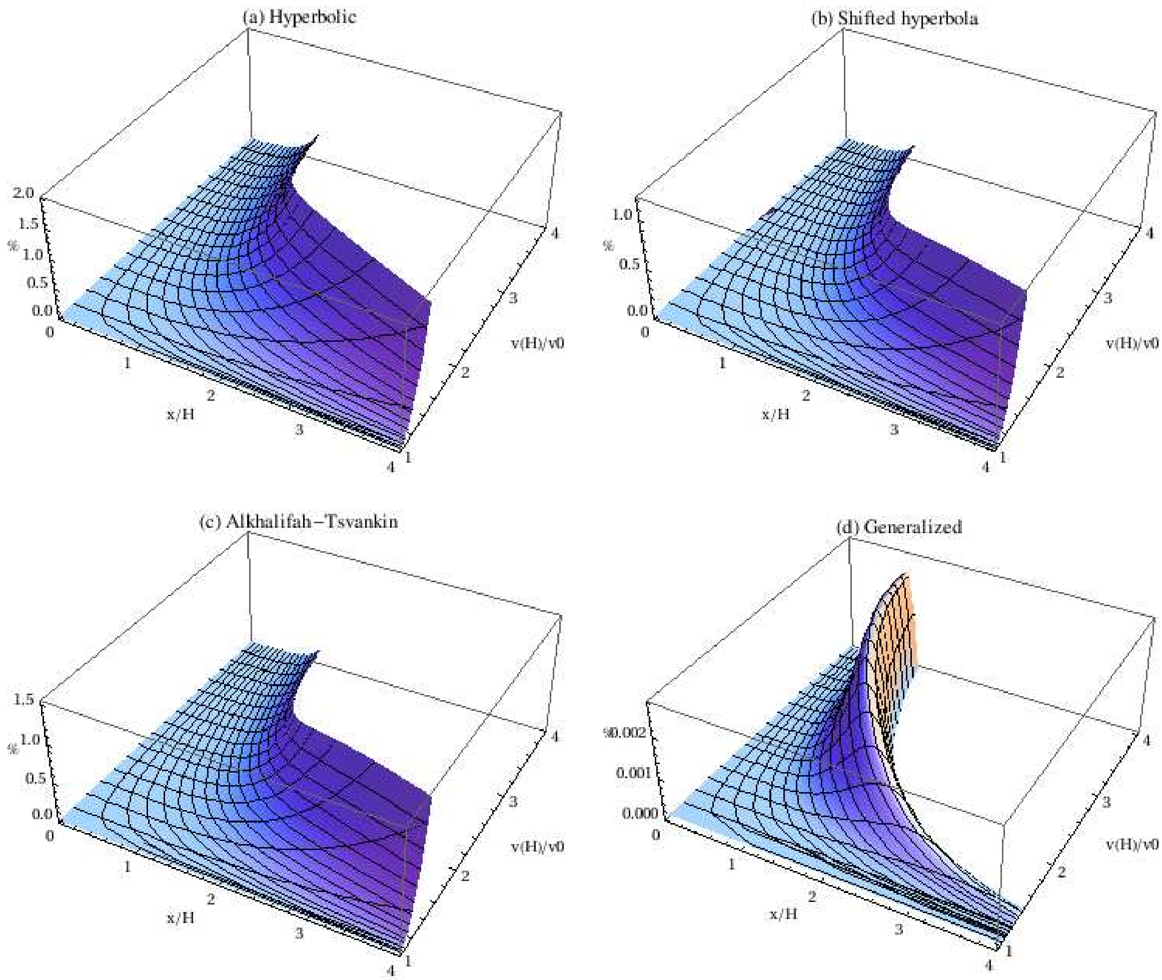

Figure 2: Relative absolute error of different traveltime approximations as a function of velocity contrast and offset/depth ratio for the case of a linear sloth model. (a) Hyperbolic approximation, (b) Shifted hyperbola approximation, (c) Alkhalifah-Tsvankin approximation, (d) Generalized nonhyperbolic approximation. The proposed generalized approximation reduces the maximum approximation error by several orders of magnitude.

happens to be exact. In this case,

$$T_\infty^2 = t_0^2 \, \frac{G}{G + \tan^2 \beta} \; , \tag{30}$$

$$P_\infty = \frac{1}{V} = \frac{1}{v \, \cos \beta} \; , \tag{31}$$

which, after substitution in equations 25-26, produce

$$B = G - \tan^2 \beta \; , \tag{32}$$

$$C = \left(G + \tan^2 \beta\right)^2 \; . \tag{33}$$

In the special case of a plane (zero curvature) reflector, $G = 0$, and the generalized approximation reduces to a hyperbolic form. In the special case of a diffraction point (infinite curvature), $G = 1$, and the generalized approximation reduces to the double-square-root equation 17. In both of those cases, as well as in the case of a hyperbolic reflector, the generalized approximation is simply exact.

Figure 3 shows a comparison between different approximations for the case of a circular reflector, analyzed in Appendix D. As in the other examples, the proposed five-parameter generalized approximation brings an improvement in accuracy in several orders of magnitude in comparison with the three-parameter approximations.

Homogeneous VTI layer

Our next analytical example is a horizontal reflector in a homogeneous VTI (vertically transverse isotropic) medium. As derived in Appendix E, the approximation coefficients, under the assumption of the acoustic approximation of Alkhalifah (1998), take the form

$$A = -4\,\eta \; , \tag{34}$$

$$B = \frac{1 + 8\,\eta + 8\,\eta^2}{1 + 2\,\eta} \; , \tag{35}$$

$$C = \frac{1}{(1 + 2\,\eta)^2} \; . \tag{36}$$

Equation 2 with coefficients given by equations 34-36 is precisely equivalent to the traveltime approximation suggested previously by Fomel (2004). Fomel (2004) shows comparisons with alternative non-hyperbolic approximations, which demonstrate superior accuracy of equation 2 in case of strongly anisotropic media.

Numerical example

For a numerical test, we create a one-dimensional velocity model by extracting a depth column out of the anisotropic Marmousi model, created by Alkhalifah (1997a) and

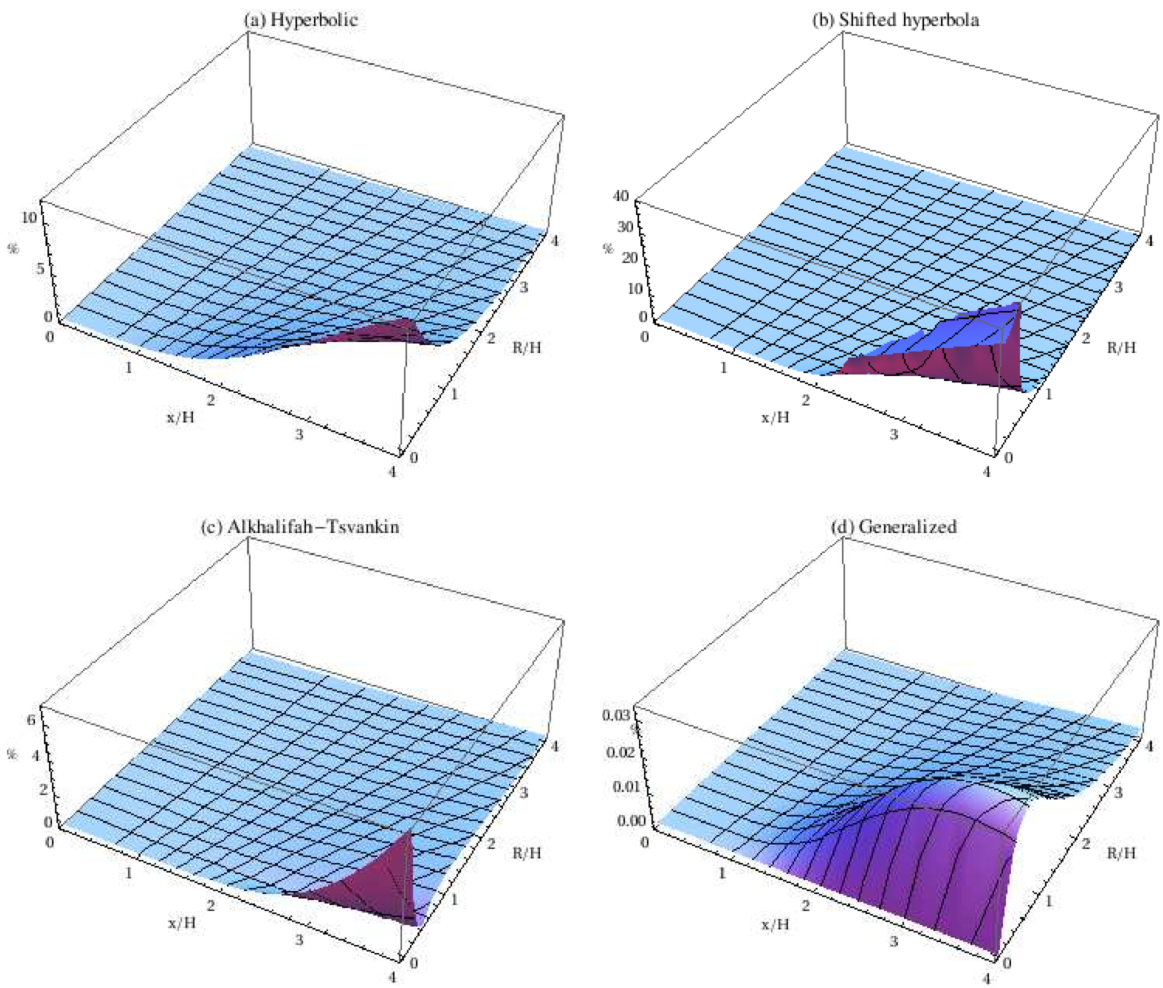

Figure 3: Relative absolute error of different traveltime approximations for the case of a circular reflector as a function of the radius/depth ratio and the offset/depth ratio. The midpoint location with respect to the center of the circle is equal to the depth of the reflector. (a) Hyperbolic approximation, (b) Shifted hyperbola approximation, (c) Alkhalifah-Tsvankin approximation, (d) Generalized nonhyperbolic approximation. The proposed generalized approximation reduces the maximum approximation error by several orders of magnitude.

shown in Figure 4. We evaluate exact reflection traveltimes by ray tracing (Figure 5). Next, we compare the exact time for different reflection rays with values predicted by different traveltime approximations. As shown in Figure 6, only the proposed generalized approximation is able to predict the true traveltime accurately over the full range of offsets. To define approximation parameters, we used equations 19-21 and 22-23 and the ray with the largest offset as the reference ray.

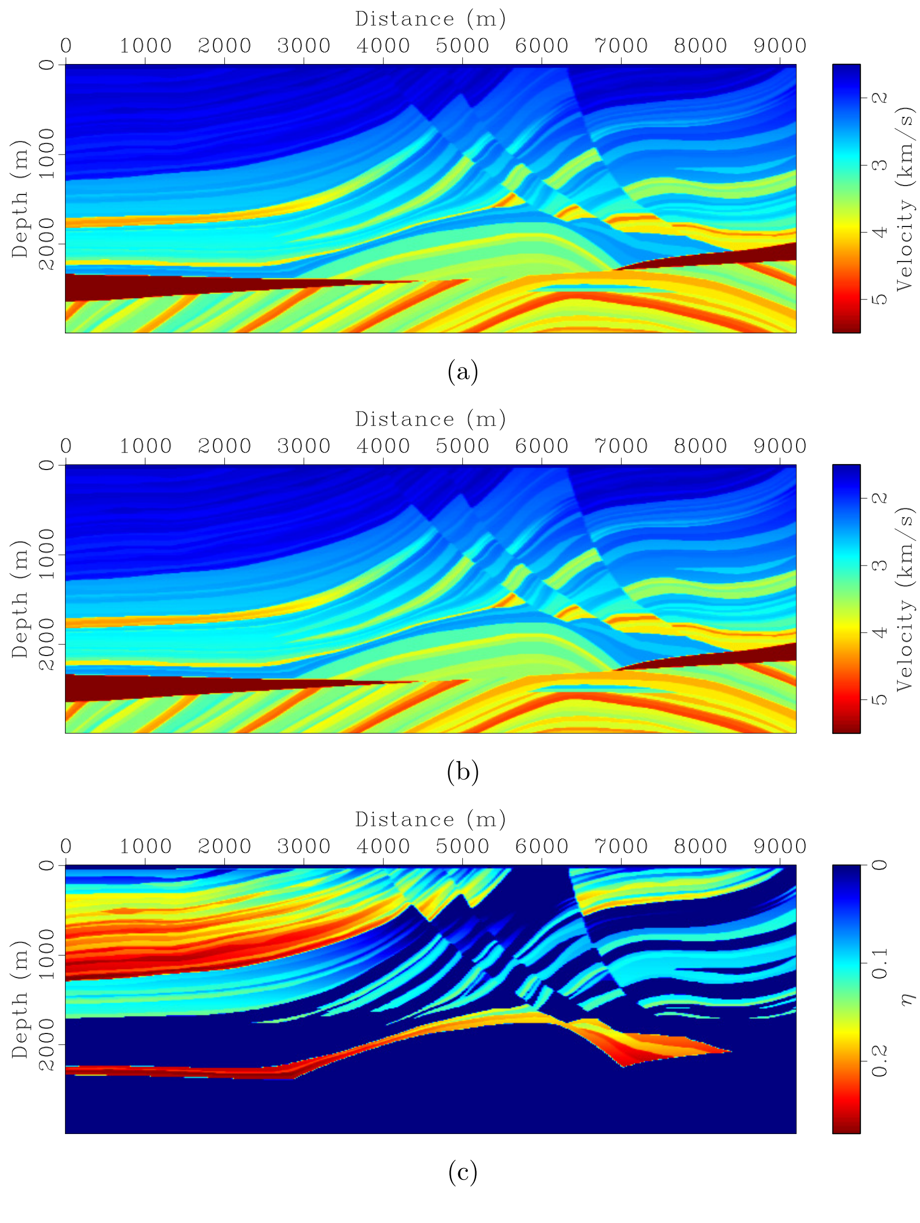

Figure 4: Anisotropic Marmousi model. (a) Vertical velocity. (b) NMO velocity. (c) η parameter.

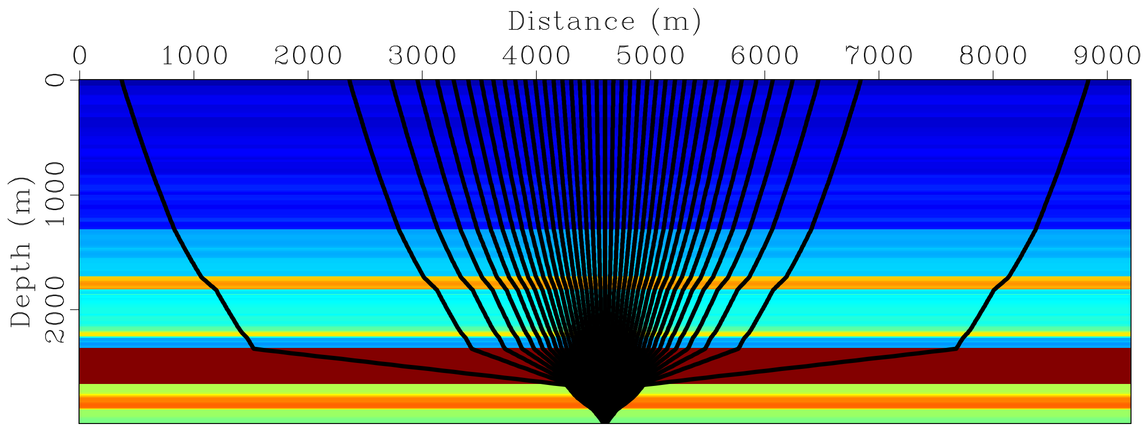

Figure 5: One-dimensional model extracted from the left column of the anisotropic Marmousi model and corresponding reflection rays.

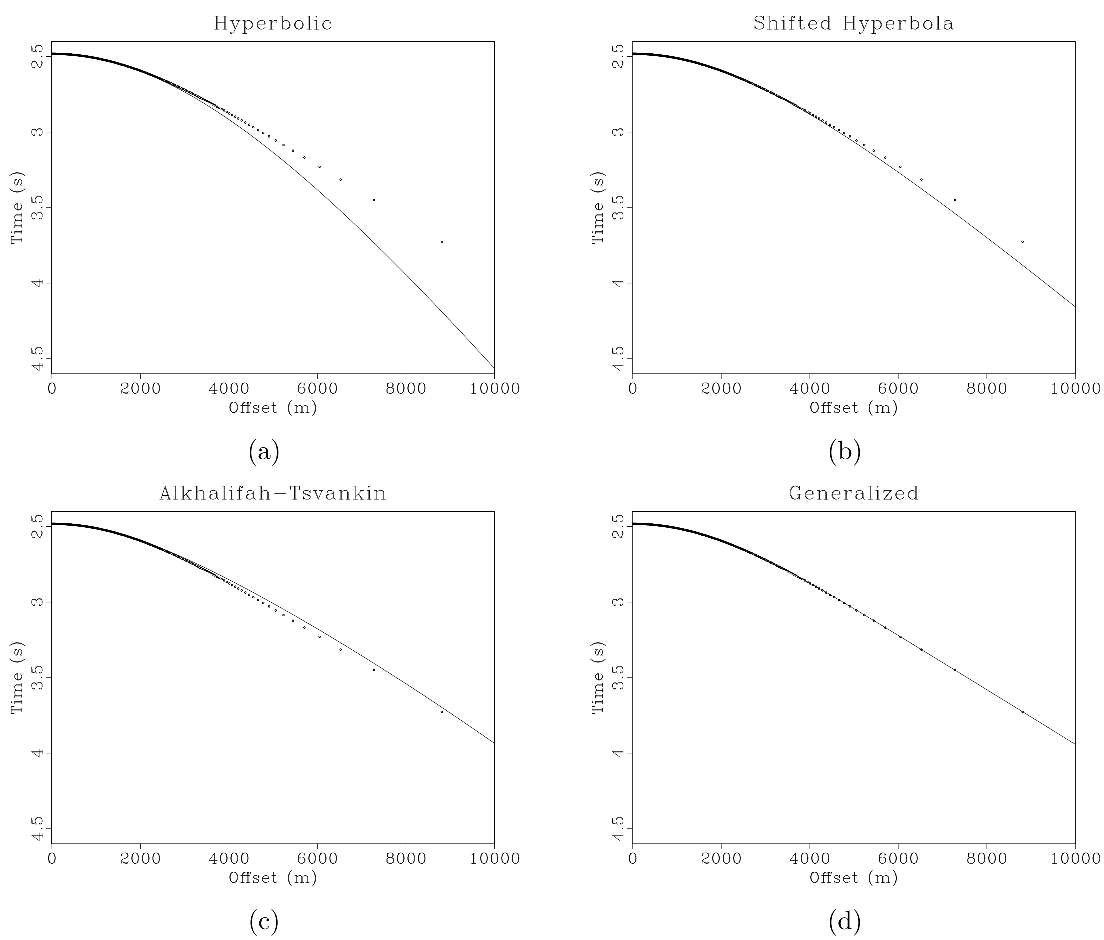

Figure 6: Exact moveout from ray tracing in the one-dimensional anisotropic Marmousi model (dots) and different approximations (solid lines). (a) Hyperbolic approximation, (b) Shifted hyperbola approximation, (c) Alkhalifah-Tsvankin approximation, (d) Generalized nonhyperbolic approximation.

DISCUSSION

Approximation is more of an art than a science. We don't have a justification for suggesting equations 1 or 2 other than pointing out that they reduce to known forms with particular choices of parameters.

The choice of a proper functional form is important for the approximation accuracy. Suppose that we try to replace the five-parameter approximation in equation 2 with the four-parameter equation

$$t^2(x) \approx t_0^2 + \frac{x^2}{v^2} + \frac{A\,x^4}{v^4\,\left(2\,t_0^2 + D\,\frac{x^2}{v^2}\right)}\;, \qquad (37)$$

where $D = B + \sqrt{C}$. Equation 37 has the same behavior as equation 2 at small offsets and the same asymptote as x approaches infinity. However, its accuracy is not nearly as spectacular (Figure 7).

A proper selection of the reference ray for equations 22 and 23 is also important for approximation accuracy. If this ray is taken not at the largest possible offset, the accuracy will deteriorate. As an extreme example, suppose that we try to define B and C by fitting subsequent terms of the Taylor expansion 18 near the zero offset rather than the behavior of the approximation at large offsets. Figure 8 shows the result for the case of a linear sloth model: the approximation is more accurate than alternatives (shown in Figure 2) but not nearly as accurate as the generalized approximation fitted at the critical offset.

Possible extensions of this work may include nonhyperbolic approximations for diffraction traveltimes (for use in prestack time migration) and reflection surfaces (for use in common-reflection-surface methods) as well as approximations for anisotropic phase and group velocities in ray tracing and wave extrapolation.

CONCLUSIONS

We propose a five-parameter nonhyperbolic moveout approximation that generalizes the classic two-parameter hyperbolic approximation as well as some known three-parameter approximations. We propose a method for selecting the approximation parameters, which involves only two rays: the normal-incident ray and one additional ray, preferably at a large offset. The special case of the additional ray being horizontal can be handled as well.

A comparison with the classic hyperbolic approximation, the shifted hyperbola approximation and the Alkhalifah-Tsvankin approximation for analytical and numerical isotropic and transversely isotropic models shows that the proposed generalized nonhyperbolic approximation can bring an improvement of several orders of magnitude in approximation accuracy. Based on these experiments, we claim that, for

Figure 7: Relative absolute error of Padé approximation in equation 37 as a function of velocity contrast and offset/depth ratio for the case of a linear sloth model. Compare with Figure 2.

Figure 8: Relative absolute error of the generalized approximation fitted to the zero offset as opposed to the critical offset. Compare with Figure 2.

many practical purposes, the proposed approximation can be used in place of the exact moveout.

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APPENDIX A: LINEAR VELOCITY MODEL

The linear velocity model is defined by

$$V(z) = V_0\,(1 + g\,z) \tag{A-1}$$

where g is the velocity gradient and V_0 is velocity at zero depth.

The reflection traveltime can be expressed in an analytical form as a function of offset (Slotnick, 1959)

$$t(x) = \frac{2\,H}{V_0\,(r-1)}\,\text{arccosh}\left[1 + \frac{(r-1)^2}{2\,r}\,\left(1 + \frac{x^2}{4\,H^2}\right)\right]\,, \tag{A-2}$$

where H is the depth of the reflector, and $r = V(H)/V_0$ is the ratio of velocity at the bottom and the top of the model. The traveltime parameters are given by

$$t_0 = \frac{2\,H}{V_0}\,\frac{\ln r}{r-1}\,, \tag{A-3}$$

$$v^2 = V_0^2\,\frac{r^2-1}{2\,\ln r}\,, \tag{A-4}$$

$$A = \frac{1}{2}\,\left(1 - \frac{r^2+1}{r^2-1}\,\ln r\right)\,. \tag{A-5}$$

This model has maximum (critical) offset and traveltime that are defined by

$$X = 2\,H\,\sqrt{\frac{r+1}{r-1}}\,, \tag{A-6}$$

$$T = \frac{2\,H}{V_0}\,\frac{\text{arccosh}\,r}{r-1}\,. \tag{A-7}$$

Substituting equations A-6 and A-7 into equations 22-23 and also using the expressions for traveltime parameters A-3, A-4, and A-5 results in complicated but analytical expressions for additional parameters B and C.

APPENDIX B: LINEAR SLOTH MODEL

The linear sloth model is defined by

$$\frac{1}{V^2(z)} = \frac{1}{V_0^2}\,(1 + G\,z)\;. \tag{B-1}$$

where G is the sloth gradient and V_0 is velocity at zero depth.

The equation for traveltime can be computed analytically, as follows (Červený, 2001):

$$t^2(x) = t_0^2 + \frac{x^2}{v^2} - \frac{x^4\,(r^2-1)^2\,(2\,Q+1)}{144\,Q^3\,H^2\,v^2\,(Q+1)^2}\;, \tag{B-2}$$

where H is the depth of the reflector, $r = V(H)/V_0$ is the ratio of velocity at the bottom and the top of the model and

$$Q = \sqrt{1 - \frac{x^2\,(r^2-1)^2}{16\,r^2\,H^2}}\;.$$

The main traveltime parameters are given by

$$t_0 = \frac{4\,H}{3\,V_0}\,\frac{1+r+r^2}{r\,(r+1)}\;, \tag{B-3}$$

$$v^2 = V_0^2\,\frac{3\,r^2}{1+r+r^2}\;, \tag{B-4}$$

$$A = -\frac{(r-1)^2}{6\,r}\;. \tag{B-5}$$

The maximum offset and traveltime are defined by

$$X = \frac{4\,H}{\sqrt{r^2-1}}\;, \tag{B-6}$$

$$T = \frac{4\,H}{3\,V_0}\,\frac{r^2+2}{r\,\sqrt{r^2-1}}\;. \tag{B-7}$$

Substituting equations B-6 and B-7 into equations 22-23 and using the expressions for traveltime parameters B-3, B-4, and B-5 results in the following analytical expressions for additional parameters B and C:

$$B = -\frac{(r-1)^2\,(1+r+r^2)}{2\,r\,(r+2)\,(2\,r+1)}\;, \tag{B-8}$$

$$C = -\frac{(r-1)^4\,(1+r+r^2)^2}{3\,r\,(r+2)\,(2\,r+1)^2}\;. \tag{B-9}$$

APPENDIX C: REFLECTION FROM A HYPERBOLIC REFLECTOR IN A HOMOGENEOUS VELOCITY MODEL

In this appendix, we derive an analytical expression for reflection traveltime from a hyperbolic reflector in a homogeneous velocity model (Figure C-1). Similar derivations apply to an elliptic reflector and were used previously in the theory of dip moveout, offset continuation, and non-hyperbolic common-reflection surface (Stovas and Fomel, 1996; Fomel, 2003; Fomel and Kazinnik, 2009).

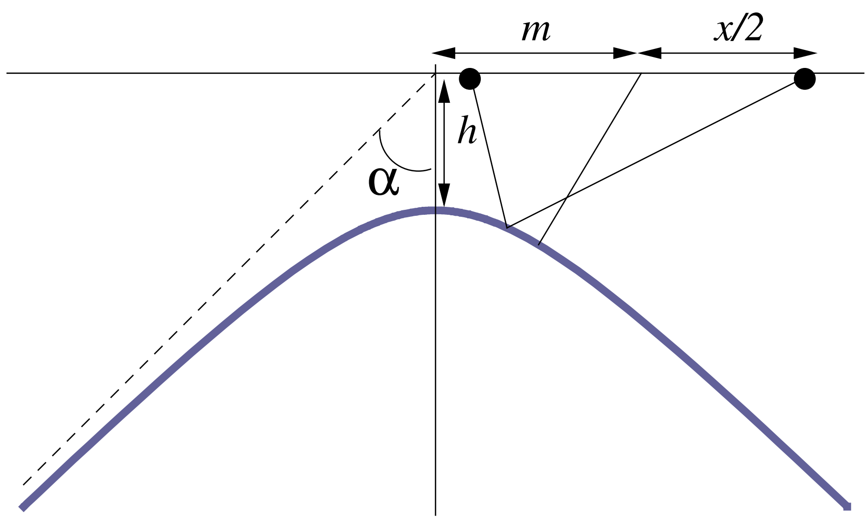

Figure C-1: Reflection from a hyperbolic reflector in a homogeneous velocity model (a scheme).

Consider the source point x_s and the receiver point x_r at the surface $z = 0$ above a 2-D constant-velocity medium and a hyperbolic reflector defined by the equation

$$z(x) = \sqrt{h^2 + x^2\,\tan^2\alpha}\,. \tag{C-1}$$

The reflection traveltime as a function of the reflection point location y is

$$t = \frac{\sqrt{(x_s-y)^2+z^2(y)}+\sqrt{(x_r-y)^2+z^2(y)}}{V}\,. \tag{C-2}$$

According to Fermat's principle, the traveltime should be stationary with respect to the reflection point y:

$$0 = \frac{\partial t}{\partial y} = \frac{y - x_s + y\,\tan^2\alpha}{V\,\sqrt{(x_s-y)^2+h^2+y^2\,\tan^2\alpha}} + \frac{y - x_r + y\,\tan^2\alpha}{V\,\sqrt{(x_r-y)^2+h^2+y^2\,\tan^2\alpha}}\,. \tag{C-3}$$

Putting two terms in equation C-3 on different sides of the equation, squaring them, and reducing their difference to a common denominator, we arrive at the equation

$$0 \;=\; \left[\frac{y}{\cos^2\alpha} - x_s\right]^2\,\left[(x_r-y)^2+h^2+y^2\,\tan^2\alpha\right]$$

$$- \left[\frac{y}{\cos^2\alpha} - x_r\right]^2 \left[(x_s - y)^2 + h^2 + y^2\,\tan^2\alpha\right] \qquad \text{(C-4)}$$

which simplifies to the following quadratic equation with respect to y:

$$y^2\,(x_s + x_r)\,\tan^2\alpha - 2\,y\,\left(x_s\,x_r\,\sin^2\alpha - h^2\right) - h^2\,(x_s + x_r)\,\cos^2\alpha = 0\,. \qquad \text{(C-5)}$$

The discriminant is

$$D = \left(x_s\,x_r\,\sin^2\alpha - h^2\right)^2 + h^2\,(x_s + x_r)^2\,\sin^2\alpha = (h^2 + x_s^2\,\sin^2\alpha)\,(h^2 + x_r^2\,\sin^2\alpha)\,. \qquad \text{(C-6)}$$

Only one of the two branches of the solution

$$\begin{aligned} y &= \frac{x_s\,x_r\,\sin^2\alpha - h^2 + \sqrt{(h^2 + x_s^2\,\sin^2\alpha)\,(h^2 + x_r^2\,\sin^2\alpha)}}{(x_s + x_r)\,\tan^2\alpha} \\ &= \frac{h^2\,(x_s + x_r)\,\cos^2\alpha}{h^2 - x_s\,x_r\,\sin^2\alpha + \sqrt{(h^2 + x_s^2\,\sin^2\alpha)\,(h^2 + x_r^2\,\sin^2\alpha)}} \end{aligned} \qquad \text{(C-7)}$$

has physical meaning. Substituting equation C-7 into equation C-2, we obtain, after a number of algebraic simplifications,

$$t = \frac{\sqrt{2h^2 + x_s^2 + x_r^2 - 2\,x_s\,x_r\,\cos^2\alpha + 2\,\sqrt{(h^2 + x_s^2\,\sin^2\alpha)\,(h^2 + x_r^2\,\sin^2\alpha)}}}{V}\,. \qquad \text{(C-8)}$$

Making the variable change in equation C-8 from x_s and x_r to the midpoint and offset coordinates m and x according to $x_s = m - x/2$, $x_r = m + x/2$, we notice that this equation is exactly equivalent to equation 1 with the following definition of parameters:

$$t_0 = \frac{2\,\sqrt{h^2 + m^2\,\sin^2\alpha}}{V}\,, \qquad \text{(C-9)}$$

$$a = \frac{2 - \sin^2\alpha}{V^2}\,, \qquad \text{(C-10)}$$

$$b = \frac{\sin^2\alpha}{V^2}\,\frac{h^2 - m^2\,\sin^2\alpha}{h^2 + m^2\,\sin^2\alpha}\,, \qquad \text{(C-11)}$$

$$c = \frac{\sin^4\alpha}{V^4}\,, \qquad \text{(C-12)}$$

$$\xi = \frac{1}{2}\,. \qquad \text{(C-13)}$$

APPENDIX D: REFLECTION FROM A CIRCULAR REFLECTOR IN A HOMOGENEOUS VELOCITY MODEL

In the case of a circular (cylindrical or spherical) reflector in a homogeneous velocity model, there is no closed-form analytical solution. However, the moveout can be described analytically by parametric relationships (Glaeser, 1999).

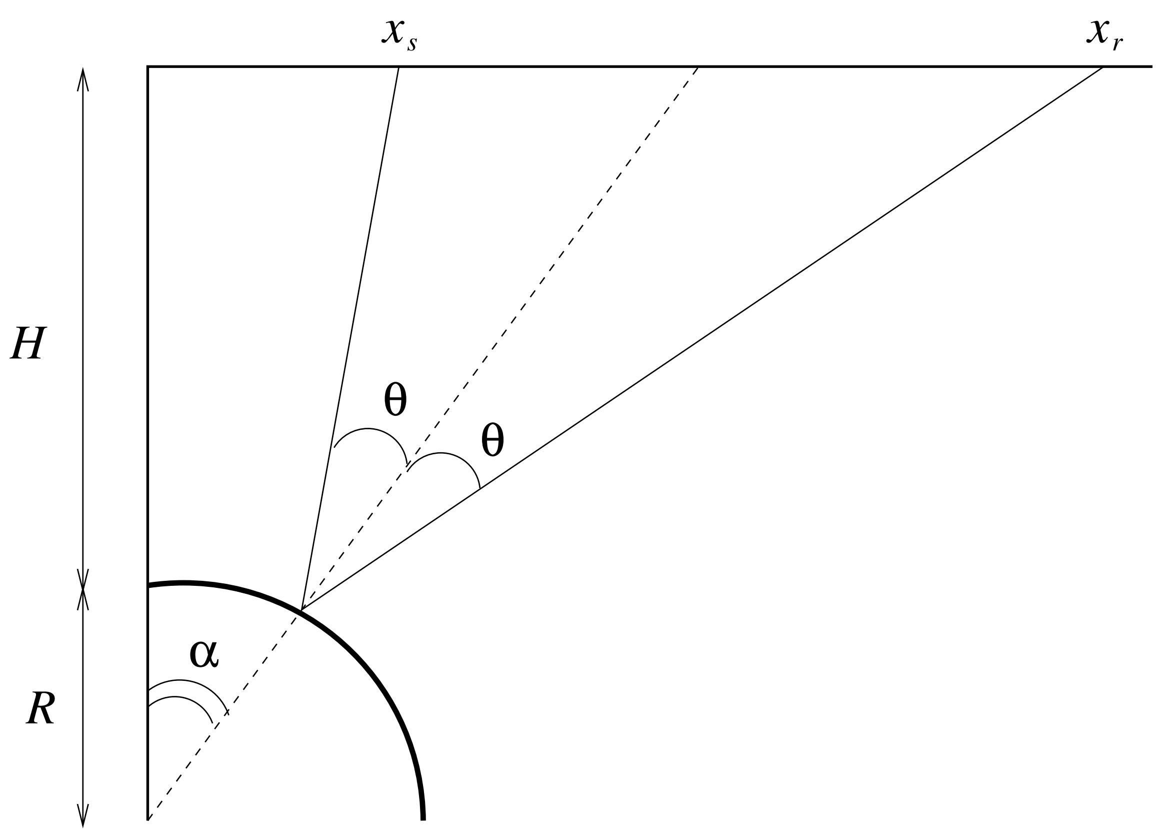

Figure D-1: Reflection from a circular reflector in a homogeneous velocity model (a scheme).

Consider the geometry of the reflection shown in Figure D-1. According to the trigonometry of the reflection triangles, the source and receiver positions can be expressed as

$$x_s = R\,\sin\alpha + (H + R - R\,\cos\alpha)\,\tan(\alpha - \theta)\;, \tag{D-1}$$
$$x_r = R\,\sin\alpha + (H + R - R\,\cos\alpha)\,\tan(\alpha + \theta)\;, \tag{D-2}$$

where R is the reflector radius, H is the minimum reflector depth, α is the reflector dip angle at the reflection point, and θ is the reflection angle. Correspondingly, the midpoint and offset coordinates can be expressed as

$$m = \frac{x_s + x_r}{2} = R\,\sin\alpha + (H + R - R\,\cos\alpha)\,\frac{\cos\alpha\,\sin\alpha}{\cos^2\theta - \sin^2\alpha}\;, \tag{D-3}$$
$$x = x_r - x_s = 2\,(H + R - R\,\cos\alpha)\,\frac{\cos\theta\,\sin\theta}{\cos^2\theta - \sin^2\alpha}\;, \tag{D-4}$$

and the reflection traveltime can be expressed as

$$\begin{aligned} t &= \frac{H + R - R\,\cos\alpha}{V}\left[\frac{1}{\cos(\alpha-\theta)} + \frac{1}{\cos(\alpha+\theta)}\right] \\ &= 2\,\frac{H + R - R\,\cos\alpha}{V}\,\frac{\cos\alpha\,\cos\theta}{\cos^2\theta - \sin^2\alpha}\;, \end{aligned} \tag{D-5}$$

where V is the medium velocity. Expressing the reflection angle θ from equation D-3 and substituting it into equations D-4 and D-5, we obtain a pair of parametric equations

$$x^2(\alpha) = 4\,\frac{[m\,\cos\alpha - (H+R)\,\sin\alpha]\,[m\,\sin\alpha + (H+R)\,\cos\alpha - R]}{\cos\alpha\,\sin\alpha}\;, \tag{D-6}$$
$$t^2(\alpha) = \frac{4}{V^2}\,\frac{(m - R\,\sin\alpha)\,[m\,\sin\alpha + (H+R)\,\cos\alpha - R]}{\sin\alpha}\;, \tag{D-7}$$

which define the exact reflection moveout for the case of a circular reflector in a homogeneous medium.

The connection with parameters of equations 27-29 is given by

$$L = \sqrt{m^2 + (H+R)^2} - R\;, \tag{D-8}$$
$$\cos\beta = \frac{H+R}{\sqrt{m^2 + (H+R)^2}}\;, \tag{D-9}$$
$$G = \frac{L}{L+R} = 1 - \frac{R}{\sqrt{m^2 + (H+R)^2}}\;. \tag{D-10}$$

The behavior of the moveout at infinitely large offsets is controlled by $P_\infty = 1/V$ and

$$T_\infty = \frac{2\,H}{V} = t_0\,\frac{G + \cos\beta - 1}{G\,\cos\beta}\;. \tag{D-11}$$

After substitution in equations 25-26, we obtain somewhat complicated but analytical expressions for parameters B and C.

APPENDIX E: HOMOGENEOUS VTI MODEL

According to the acoustic approximation of Alkhalifah (1998), one can use the following parametric equations to define the traveltime-offset relationship in a homogeneous VTI model:

$$x(p) = \frac{2\,H}{v_z}\,\frac{p\,v^2}{(1-2\,\eta\,p^2\,v^2)^2\,\sqrt{1-\frac{p^2\,v^2}{1-2\,\eta\,p^2\,v^2}}}\;, \tag{E-1}$$

$$t(p) = \frac{2\,H}{v_z}\,\frac{(1-2\,\eta\,p^2\,v^2)^2+2\,\eta\,p^4\,v^4}{(1-2\,\eta\,p^2\,v^2)^2\,\sqrt{1-\frac{p^2\,v^2}{1-2\,\eta\,p^2\,v^2}}}\;, \tag{E-2}$$

where p is the ray parameter, H is the depth of the reflector, v_z is the vertical velocity, v is the NMO velocity, and η is the dimensionless parameter introduced by Alkhalifah and Tsvankin (1995).

At small offsets, the homogeneous VTI traveltime behaves as

$$t^2(x) \approx t_0^2 + \frac{x^2}{v^2} - \frac{2\,\eta\,x^4}{t_0^2\,v^4}\;, \tag{E-3}$$

which allows us to define $A = -4\,\eta$ according to equation 18.

At large offsets, the homogeneous VTI traveltime behaves as

$$t^2(x) \approx t_0^2\,(1+2\,\eta) + \frac{x^2}{v^2\,(1+2\,\eta)}\;. \tag{E-4}$$

Comparing with equation 24, we note that $T_\infty = t_0\,\sqrt{1+2\,\eta}$ and $P_\infty = 1/(v\,\sqrt{1+2\,\eta})$. Substituting into equations 25-26, we derive the coefficients B and C to be

$$B = \frac{1+8\,\eta+8\,\eta^2}{1+2\,\eta}\;, \tag{E-5}$$

$$C = \frac{1}{(1+2\,\eta)^2}\;. \tag{E-6}$$

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